A Theory of Hanoi Omega Automata and Games

Saina Sunny, *saina.sunny@ulb.be*
Université libre de Bruxelles



Emmanuel Filiot
efiliot@gmail.com
Université libre de Bruxelles



Guillermo Pérez
guillermo.perez@uantwerpen.be
University of Antwerp



Allen Joseph
allenmaniyakupara@gmail.com
Université libre de Bruxelles

24 April, 2026

CFV Seminar, ULB

Formal Verification of Reactive Systems

Model Checking

- Given a property φ, model $\mathcal{M}$.
- Does $\mathcal{M} \models \varphi$?

Synthesis

- Given a property φ.
- Can you construct a model $\mathcal{M}$ s.t. $\mathcal{M} \models \varphi$?

- Property is encoded in an infinite word automaton.

Encoding Property

$$\text{Atomic Propositions} = \{\mathsf{request}, \mathsf{grant}\}$$

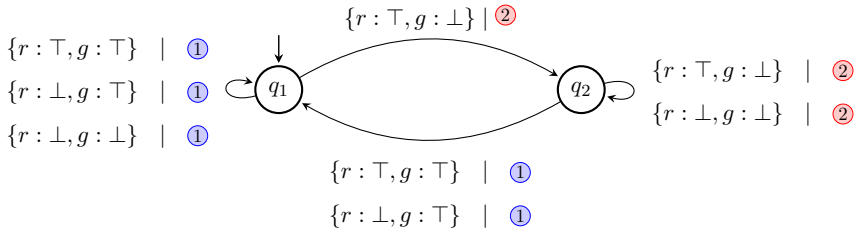
$$\text{Property: } \mathsf{Globally}(\mathsf{request} \implies \mathsf{Eventually}(\mathsf{grant}))$$

Encoding Property

$$\text{Atomic Propositions} = \{\mathsf{request}, \mathsf{grant}\}$$

$$\text{Property: } \mathsf{Globally}(\mathsf{request} \implies \mathsf{Eventually}(\mathsf{grant}))$$

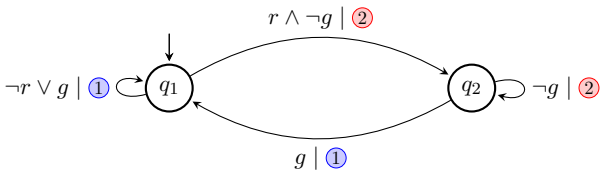
Alphabet: Valuations of propositions ($\{\mathsf{request}, \mathsf{grant}\} \mapsto \{\top, \bot\}$)



Acceptance: color ① occurs infinitely often.

Hanoi Omega Automata (HOA)

- HOA is a standard format for encoding ω-regular automata.



Acceptance: $\mathsf{Inf}(\{(1)\})$.

- Boolean formulas on transitions.
- Input word: sequence of valuations $v_1 v_2 \cdots$ where $v_i : \mathsf{Propositions} \mapsto \{\top, \bot\}$.
- Acceptance: Boolean formula over predicates $\mathsf{Inf(C)}$ and $\mathsf{Fin(C)}$ for any $C \subseteq \mathsf{colors}$.

$$c_1 c_2 \cdots \models \mathsf{Inf(C)} \text{ if } \exists c \in C \text{ s.t. } c \text{ occurs infinitely often in } c_1 c_2 \cdots$$

$$c_1 c_2 \cdots \models \mathsf{Fin(C)} \text{ if } \forall c \in C \text{ s.t. } c \text{ occurs finitely often in } c_1 c_2 \cdots$$

ω-regular Acceptance Conditions

- Emerson-Lei: Boolean formula over predicates $\mathsf{Inf(C)}$ and $\mathsf{Fin(C)}$ for any $C \subseteq \mathsf{colors}$.

Büchi	B	$\mathsf{Inf}(B)$
co-Büchi	C	$\mathsf{Fin}(C)$
Rabin	$\{(E_1, F_1), \ldots, (E_k, F_k)\}$	$\bigvee_{i\in[k]}(\mathsf{Inf(E_i)} \wedge \mathsf{Fin(F_i)})$
Streett	$\{(E_1, F_1), \ldots, (E_k, F_k)\}$	$\bigwedge_{i\in[k]}(\mathsf{Fin(E_i)} \vee \mathsf{Inf(F_i)})$
Muller	$\{E_1, \ldots, E_k\}$	$\bigvee_{i\in[k]} \bigwedge_{c\in E_i} \mathsf{Inf}(\{\mathsf{c}\})$
Parity	max even	$\bigvee_{c\in[d], c \text{ is even}} \mathsf{Inf}(\{\mathsf{c}\}) \wedge \mathsf{Fin}(\{\mathsf{c}+1, \ldots, \mathsf{d}\})$
Reachability	R	eventually see some color in R
Safety	S	always see colors in S

Why HOA?

- Unique structural features:
  - Succinct representation : symbolic alphabet via formulas.
  - Transition-based acceptance.
  - Generalised acceptance condition.
- Standard format for ω-automata used in synthesis tools, such as

  Spot, Owl, LTL3TELA

- HOA-encoded games are standard input format for SYNTCOMP competition.

Gap: Understanding theoretical complexity of HOA-encoded automata.

Contributions

We study the complexity of following problems of Hanoi Omega Automata.

- Nonemptiness of HOA.
- Language inclusion of HOAs.
- Game solving involving deterministic[1] HOA.

[1] For every state and every valuation v, there is at most one transition whose formula is satisfied by v.

Nonemptiness of HOA

Given an HOA $\mathcal{A}$ and an accepting condition Acc,

Does there $\exists$ an infinite word accepted by $\mathcal{A}$?

$$L_{Acc}(\mathcal{A}) \neq \emptyset?$$

Theorem

The nonemptiness problem for HOA with an Emerson-Lei acceptance condition is **NP***-complete. It is* **NP***-hard already for deterministic safety and reachability HOA automata.*

Nonemptiness of HOA

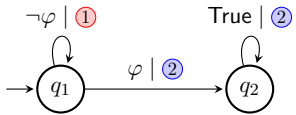
Theorem

The nonemptiness problem for HOA with an Emerson-Lei acceptance condition is **NP***-complete. It is* **NP***-hard already for deterministic safety and reachability HOA automata.*

- **NP**-hardness: Reduction from SAT.

Boolean formula φ is satisfiable $\iff L_{Acc}(\mathcal{A}) \neq \emptyset$ with $Acc = \mathsf{Inf}\{2\}$.

HOA $\mathcal{A}$:



Nonemptiness of HOA

Theorem

The nonemptiness problem for HOA with an Emerson-Lei acceptance condition is **NP***-complete. It is* **NP***-hard already for deterministic safety and reachability HOA automata.*

- **NP**-upperbound: Given an HOA $\mathcal{A}$ with acceptance *Acc*.
- If $\mathcal{A}$ has an accepting run, it has a polynomial-size lasso-shaped accepting run preserving fin. and inf. occurring colors.
- Nondeterministically guess such a run and valuation for its transitions.
- Check whether valuation satisfies the formulas on transitions.
- Check that the fin. and inf. occurring colors in the run satisfy *Acc*.

Language Inclusion of HOAs

Given HOAs $\mathcal{A}$, $\mathcal{B}$ (over same atomic propositions) and an accepting condition Acc,

Every infinite word accepted by $\mathcal{A}$ is also accepted by $\mathcal{B}$?

$$L_{Acc}(\mathcal{A}) \subseteq L_{Acc}(\mathcal{B})?$$

Theorem

The language inclusion problem of HOAs is

- **PSPACE**-*complete for Muller, Rabin, Streett, parity, Büchi, coBüchi, reachability and safety acceptance.*
- **EXPSPACE**-*complete for Emerson-Lei acceptance.*

Hanoi Omega Games

A two-player Hanoi omega game is a tuple

$$\mathcal{G} = (\mathcal{A}, \mathcal{P}_{\mathsf{in}}, \mathcal{P}_{\mathsf{out}}, Acc) \text{ where}$$

- $\mathcal{A}$ is a complete and deterministic[2] HOA over propositions $\mathcal{P}_{\mathsf{in}} \cup \mathcal{P}_{\mathsf{out}}$.
- Acc is a winning (accepting) condition of $\mathcal{A}$.

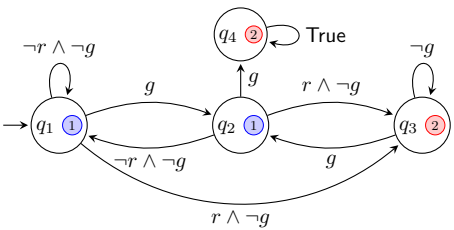
[2] For every state and every valuation v, there is exactly one transition whose formula is satisfied by v.

Hanoi Omega Games

A two-player Hanoi omega game is a tuple

$$\mathcal{G} = (\mathcal{A}, \mathcal{P}_{\mathsf{in}} = \{r\}, \mathcal{P}_{\mathsf{out}} = \{g\}, \mathsf{Inf}\{\textcircled{1}\})$$

$$G(r \to F(g)) \wedge G(g \to X(\neg g))$$

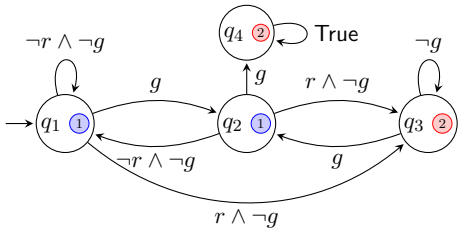


- The game is played between Player In (environment) and Player Out (controller).
- Player In controls propositions $\mathcal{P}_{\mathsf{in}}$ and Player Out controls propositions $\mathcal{P}_{\mathsf{out}}$
- The Player Out (controller) aims to satisfy the *Acc*.

Hanoi Omega Games

A two-player Hanoi omega game is a tuple

$$\mathcal{G} = (\mathcal{A}, \mathcal{P}_{\mathsf{in}} = \{r\}, \mathcal{P}_{\mathsf{out}} = \{g\}, \mathsf{Inf}\{1\})$$



The game starts from the initial state q_1,

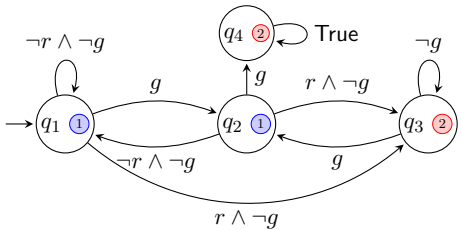
- Firstly, Player In (environment) chooses a valuation v^{in} of $\mathcal{P}_{\mathsf{in}}$,
- Then, Player Out (controller) chooses a valuation v^{out} of $\mathcal{P}_{\mathsf{out}}$.

$$q_1 \xrightarrow{\varphi} q_2 \text{ is taken if } v^{\mathsf{in}} \cup v^{\mathsf{out}} \models \varphi.$$

Hanoi Omega Games

A two-player Hanoi omega game is a tuple

$$\mathcal{G} = (\mathcal{A}, \mathcal{P}_{\mathsf{in}} = \{r\}, \mathcal{P}_{\mathsf{out}} = \{g\}, \mathsf{Inf}\{1\})$$



- A *play* is an infinite sequence of valuations $\pi = (v_1^{\mathsf{in}} \cup v_1^{\mathsf{out}})(v_2^{\mathsf{in}} \cup v_2^{\mathsf{out}}) \ldots$

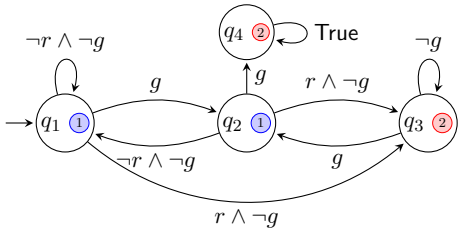
$$q_1 \xrightarrow{v_1^{\mathsf{in}} \cup v_1^{\mathsf{out}} \models \varphi_1} q_2 \xrightarrow{v_2^{\mathsf{in}} \cup v_2^{\mathsf{out}} \models \varphi_2} \cdots$$

Player Out (controller) wins if $\pi \in L_{Acc}(\mathcal{A})$, otherwise Player In (environment) wins.

Hanoi Omega Games

A two-player Hanoi omega game is a tuple

$$\mathcal{G} = (\mathcal{A}, \mathcal{P}_{\mathsf{in}} = \{r\}, \mathcal{P}_{\mathsf{out}} = \{g\}, \mathsf{Inf}\{1\})$$



- Player Out has a *winning strategy* if it can win (satisfy the Acc) irrespective of the valuations picked by Player In.
- Player In has a *winning strategy* if it can win (does not satisfy the Acc) irrespective of the valuations picked by Player Out.

HOG: Hardness

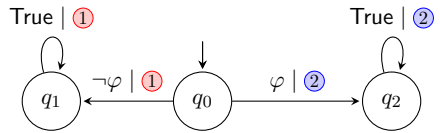
Lemma

Hanoi omega games with reachability and safety (and hence Rabin/Streett/ parity/ Büchi/ co-Büchi HOGs) winning conditions are $\mathbf{\Pi_2}$*-hard.*

- Reduction from QSAT formulas of the form $\forall\bar{x}\exists\bar{y}\varphi$ ($\mathbf{\Pi_2}$-complete).

$\forall\bar{x}\exists\bar{y}\varphi$ is satisfiable $\iff$
Player Out wins in HOG
$G_\varphi = (\mathcal{A}, \bar{x}, \bar{y}, \mathsf{Inf}\{2\})$.

$\mathcal{A}$:



[2] $\mathbf{\Pi_2} = \mathbf{coNP^{NP}}$

HOG: Upperbound

Theorem

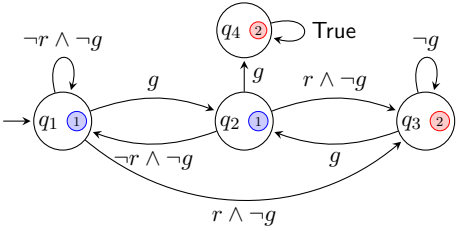
The Hanoi omega games are

- $\mathbf{\Pi_2}$*-complete for Streett/ parity/ Büchi / co-Büchi/ reachability/ safety winning conditions.*
- $\mathbf{\Sigma_3}$ *for Rabin winning condition.*
- **PSPACE***-complete for Muller and Emerson-Lei winning conditions.*

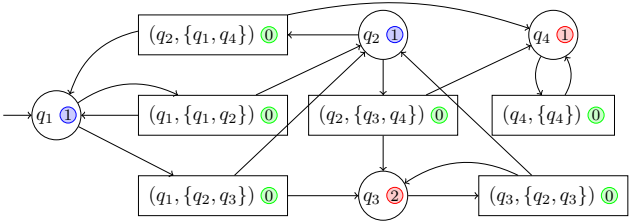
[2] $\mathbf{\Pi_2} = \mathbf{coNP^{NP}}$ and $\mathbf{\Sigma_2} = \mathbf{NP^{NP}}$ and $\mathbf{\Sigma_3} = \mathbf{NP^{\Sigma_2}}$

HOG to Classical Game

$\mathcal{G} = (\mathcal{A}, \mathcal{P}_{\mathsf{in}}, \mathcal{P}_{\mathsf{out}}, Acc)$



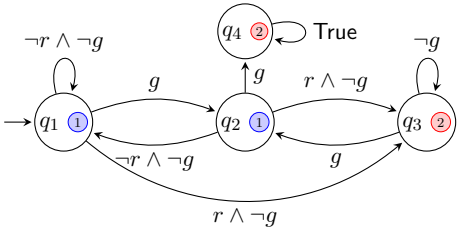
$P_{\mathcal{G}} = (V_{\mathsf{in}} = Q, V_{\mathsf{out}} = Q \times 2^Q, E, \mathcal{C}, Acc')$

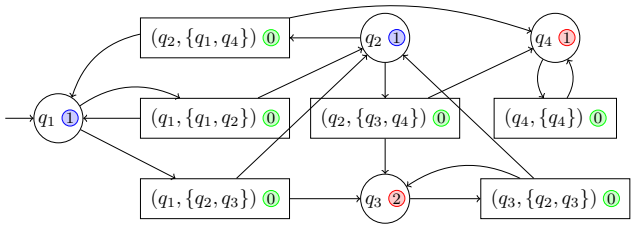


HOG to Classical Game

$\mathcal{G} = (\mathcal{A}, \mathcal{P}_{\mathsf{in}}, \mathcal{P}_{\mathsf{out}}, Acc)$

$P_{\mathcal{G}} = (V_{\mathsf{in}} = Q, V_{\mathsf{out}} = Q \times 2^Q, E, \mathcal{C}, Acc')$



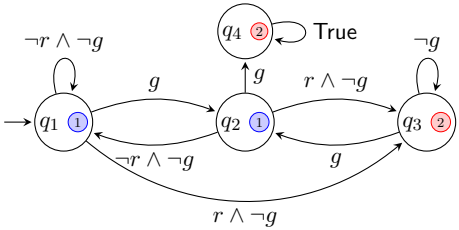


$S \subseteq Q$ is a q-good subset if $\exists v_{\mathsf{in}}$ of $\mathcal{P}_{\mathsf{in}}$ s.t. $\forall v_{\mathsf{out}}$ of $\mathcal{P}_{\mathsf{out}}$ transitions of q goes to S.

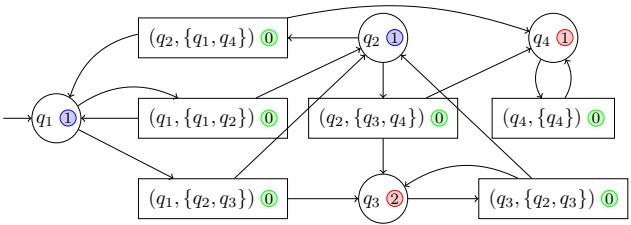
- $q \to (q, S)$ if S is a q-good subset.
- $(q, S) \to p$ if $p \in S$.

HOG to Classical Game

$\mathcal{G} = (\mathcal{A}, \mathcal{P}_{\mathsf{in}}, \mathcal{P}_{\mathsf{out}}, Acc)$



$P_{\mathcal{G}} = (V_{\mathsf{in}} = Q, V_{\mathsf{out}} = Q \times 2^Q, E, \mathcal{C}, Acc')$

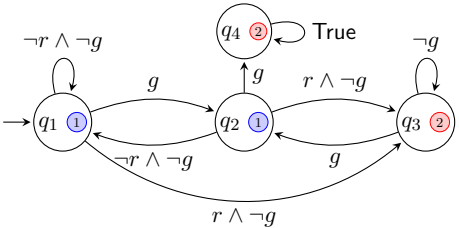


$S \subseteq Q$ is a q-good subset if $\exists v_{\mathsf{in}}$ of $\mathcal{P}_{\mathsf{in}}$ s.t. $\forall v_{\mathsf{out}}$ of $\mathcal{P}_{\mathsf{out}}$ transitions of q goes to S.

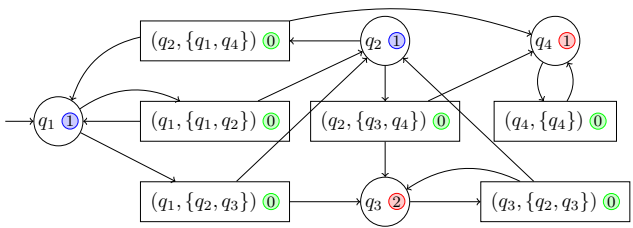
- $q \to (q, S)$ if S is a q-good subset.
- $(q, S) \to p$ if $p \in S$.
- $Acc' = \{c_1 ⓪ c_2 ⓪ \ldots \mid c_1 c_2 \ldots \in Acc\}$

HOG to Classical Game

$\mathcal{G} = (\mathcal{A}, \mathcal{P}_{\mathsf{in}}, \mathcal{P}_{\mathsf{out}}, Acc)$



$P_{\mathcal{G}} = (V_{\mathsf{in}} = Q, V_{\mathsf{out}} = Q \times 2^Q, E, \mathcal{C}, Acc')$



$$Acc' = \{c_1 ⓪ c_2 ⓪ \ldots \mid c_1 c_2 \ldots \in Acc\}$$

Remark

Player In *(resp.* Out*) has a Acc-winning strategy in* $\mathcal{G}$ *iff Player* In *(resp.* Out*) has a* Acc'*-winning strategy in* $P_{\mathcal{G}}$*.*

HOG to Classical Game

$$\mathcal{G} = (\mathcal{A}, \mathcal{P}_{\mathsf{in}}, \mathcal{P}_{\mathsf{out}}, Acc) \qquad P_{\mathcal{G}} = (V_{\mathsf{in}} = Q, V_{\mathsf{out}} = Q \times 2^Q, E, \mathcal{C}, Acc')$$

$$Acc' = \{c_1 ⓪ c_2 ⓪ \ldots \mid c_1 c_2 \ldots \in Acc\}$$

Remark

Player In *(resp.* Out*) has a Acc-winning strategy in $\mathcal{G}$ iff Player* In *(resp.* Out*) has a Acc'-winning strategy in $P_{\mathcal{G}}$.*

- Solving HOG $\mathcal{G}$ can be reduced to solving classical game $P_{\mathcal{G}}$.
- $P_{\mathcal{G}}$ is exponentially larger than that of $\mathcal{G}$.
- To obtain upperbound, $P_{\mathcal{G}}$ is not constructed explicitly.
- Idea: nondeterministically guess a "small-size" strategy of Player In or Player Out in $P_{\mathcal{G}}$ and check it is winning.

Streett HOG

- Reduce solving HOG $\mathcal{G}$ to solving game $P_{\mathcal{G}}$ (without explicitly constructing).
- In classical Streett games, Player In admits memoryless startegy.
- Guess the strategy for Player In in $P_{\mathcal{G}} : q \to v_{\mathsf{in}}, (q, S_q)$.
- Verify the strategy:
  1. construct subgraph of $P_{\mathcal{G}}$ of size $2|\mathcal{G}|$, and check it is winning for Player In.
  2. verify if chosen subset S_q is q-good $\implies$ deciding satisfiability of $\forall$-formula ($\in \mathbf{coNP}$)

$$\forall v_{\mathsf{out}} \text{ of } \mathcal{P}_{\mathsf{out}} \bigwedge_{q \xrightarrow{\varphi} p \notin S_q} \neg\varphi\big(v_q^{\mathsf{in}}, v_{\mathsf{out}}\big).$$

- Deciding whether Player In wins belongs to $\mathbf{NP}^{\mathbf{coNP}} = \mathbf{NP}^{\mathbf{NP}} = \boldsymbol{\Sigma_2}$.
- By determinancy, deciding whether Player Out wins belongs to $\boldsymbol{co\Sigma_2} = \boldsymbol{\Pi_2}$.

Rabin HOG

- Reduce solving HOG $\mathcal{G}$ to solving game $P_{\mathcal{G}}$ (without explicitly constructing).
- In classical Rabin games, Player Out admits memoryless strategy.

Lemma

If Player Out has a memoryless strategy in $P_{\mathcal{G}}$, then it has a super-memoryless strategy in $P_{\mathcal{G}}$: for each state q, there is a linear order $\leq_q$ on Q, and every (q, S), picks the $\leq_q$-minimal element in S.

- Guess the super-memoryless strategy for Player Out: for each q, a linear order $\leq_q$.
- Verify the strategy — construct graph over Q: $q \to p$ if there exists a subset S s.t. p is the $\leq_q$-minimal element of S and S is q-good. This requires querying a $\exists\forall$-formula ($\in \mathbf{\Sigma_2}$).
- Deciding whether Player Out wins belongs to $\mathbf{NP}^{\mathbf{\Sigma_2}} = \mathbf{\Sigma_3}$

HOG: Complexity

Theorem

The Hanoi omega games are

- $\mathbf{\Pi_2}$*-complete for Streett/ parity/ Büchi / co-Büchi/ reachability/ safety winning conditions.*
- $\mathbf{\Sigma_3}$ *for Rabin winning condition.*
- **PSPACE***-complete for Muller and Emerson-Lei winning conditions.*

[2] $\mathbf{\Pi_2} = \mathbf{coNP^{NP}}$ and $\mathbf{\Sigma_2} = \mathbf{NP^{NP}}$ and $\mathbf{\Sigma_3} = \mathbf{NP^{\Sigma_2}}$

Summary

Acceptance	**Emptiness**	**Language Inclusion**	**Game Solving**
Reachability / Safety / Büchi / co-Büchi / Parity / Streett	**NP**-complete	**PSPACE**-complete	Π_2-complete
Rabin	**NP**-complete	**PSPACE**-complete	Σ_3 and Π_2-hard
Muller	**NP**-complete	**PSPACE**-complete	**PSPACE**-complete
Emerson–Lei	**NP**-complete	**EXPSPACE**-complete	**PSPACE**-complete

Table 1: Complexity of decision problems for Hanoi-Omega Automata.

Thank you :-)