Distance and Conjugacy of Word Transducers

Thesis Summary

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A. Comparing Transducers

This thesis presents a nuanced approach to distinguishing between two word-to-word functions defined by transducers, moving beyond the binary concept of equivalence. A finite state transducer is a computational model that reads an input word and produces an output word using finite memory. Examples of transducers include spell checkers, grammatical tools, and speech recognition systems. While automata accept set of input words, transducers extends this by associating output words to each input word, thereby defining a relation between input and output words. Formally, if A is the input alphabet and B is the output alphabet, a transducer defines a relation over $A^* \times B^*$, known as a *rational relation*. If the relation is a graph of a function, then it is called *rational functions*. Sequential functions are strict subclass of rational functions, defined as the functions recognised by input-deterministic finite state transducers.

Given two transducers, a natural question is: how similar are they? The traditional approach is testing equivalence of transducers that determines whether the relations defined by the transducers are identical. Equivalence checking is a well-studied problem in the literature and is known to be undecidable in general [8]. But for transducers that realise functions, checking their equivalence is decidable [3] and shown to be PSPACE-complete [9].

We generalise the boolean comparison to a quantitative setting by assigning a value to a pair of transducers that indicate how different they are from each other. A metric on words over the alphabet B is a function $d : B^* \times B^* \to \mathbb{R} \cup \{\infty\}$ such that for any words u, v and w in B^*, $d(u, v) = 0 \iff u = v$, $d(u, v) = d(v, u)$ and $d(u, v) \leq d(u, w) + d(w, v)$. We lift a word metric to the class of word-to-word functions. Given a metric d on words, we define the distance between two functions f, g with the same input domain, to be the supremum of $d(f(u), g(u))$ for all words u in their domain. If their domains are different, the distance is infinite. It can be verified that d is a metric on functions. The distance $d(\mathcal{T}_1, \mathcal{T}_2)$ between two functional transducers $\mathcal{T}_1$ and $\mathcal{T}_2$ is the distance between the (rational) functions defined by them. The value of $d(T_1, T_2)$ is an upperbound on the distance between the outputs of T_1 and T_2 on any input.

An obvious question is whether we can compute the distance for relevant metrics. A notable class of metrics is edit distances. The *edit distance* between two words is the minimum number of edit operations — such as insertion, deletion, or substitution — required to rewrite one word to another if possible, and infinite otherwise. Some common edit distances are given in Table 1. Our main result is summarised as follows.

▶ **Theorem 1** ([2]). *The distance between rational functions w.r.t. a metric given in Table 1 is computable.*

Further, we introduce two generalisations of the notion of distance between functions, namely, *diameter* of a relation and *index* of a relation in the composition closure of another.

The diameter of a relation w.r.t. a metric is defined to be the supremum of the distance of every pair in the relation. This is studied for rational relations when distance over words is measured by their length difference [7]. We extend the result to edit distances as follows.

Edit Distance	Permissible Operations
Hamming	letter-to-letter substitutions
Conjugacy	left and right cyclic shifts
Transposition	swapping adjacent letters
Longest Common Subsequence	insertions and deletions
Levenshtein	insertions, deletions, and substitutions
Damerau-Levenshtein	insertions, deletions, substitutions and adjacent transpositions

Table 1 Edit Distances

▶ **Theorem 2** ([2]). *The diameter of a rational relation w.r.t. a metric given in Table 1 is computable.*

The *index* of a relation R in the composition closure of a relation S is defined to be the smallest integer k such that R is contained in the k-fold composition of S. If $k < \infty$, then R is said to have the *finite index property* in the composition closure of S. We show that the finite index property is undecidable for arbitrary rational relations. A rational relation is *metrizable* w.r.t. a metric d if the graph of the relation defines a distance equivalent to d upto boundedness. We have the following result for metrizable relations.

▶ **Theorem 3** ([2]). *The index of a rational relation in the composistion closure of a* metrizable *relation w.r.t. a metric given in Table 1 is computable.*

The computability of distance, diameter and index relies on a combinatorial result: checking whether a rational relation is conjugate. Loosely speaking, the computability rests on deciding conjugacy of a rational relation defined by the strongly connected components of the cartesian product of the two transducers.

B. Conjugacy of a Rational Relation

A pair of words is conjugate if they are cyclic shifts of each other. For instance, the pair of words $(listen, enlist)$ is conjugate, while $(loop, pool)$ is not. A relation is conjugate if every pair in the relation is conjugate. We address the decidability of conjugacy of rational relations: Given a rational relation, does it contain only conjugate pairs? We have the following result.

▶ **Theorem 4** ([1]). *Conjugacy of rational relations is decidable.*

The decidability of conjugacy relies on the notion of a *common witness* of a relation. This is inherited from Lyndon-Schützenberger's theorem [10] characterising conjugacy of two words — a pair of words (u, v) is conjugate if and only if there exists a word z such that $uz = zv$. By symmetry of conjugacy, there also exists a word z such that $zu = vz$. A word z is a *common witness* of a relation R if either for all $(u, v) \in R$, $uz = zv$, or for all $(u, v) \in R$, $zu = vz$.

We give the following characterisations for conjugacy of a set of pairs of words, which is a generalisation of Lyndon-Schützenberger theorem.

▶ **Theorem 5** ([1]). *Let G be an arbitrary set of pairs of words. The following are equivalent.*

1. *G^* is conjugate.*

2. *G^* has a common witness z.*
3. *G has a common witness z.*
4. *Roots of G has a common witness z.*

▶ **Theorem 6** ([1]). *Let $G = (u_0, v_0){G_1}^*(u_1, v_1)$ where G_1 is an arbitrary sets of pairs of words, and $(u_0, v_0), (u_1, v_1)$ are arbitrary pairs of words. The following are equivalent.*

1. *G is conjugate.*
2. *$G_1 \cup \{(u_1 u_0, v_1 v_0)\}$ has a common witness.*
3. *G has a common witness.*

In the above theorem, the common witness of G is obtained by a word equation involving u_0, u_1, v_0, v_1 and a common witness of the set $G_1 \cup \{(u_1 u_0, v_1 v_0)\}$.

▶ **Theorem 7** ([1]). *Let $G = (u_0, v_0){G_1}^*(u_1, v_1) \cdots (u_{k-1}, v_{k-1}){G_k}^*(u_k, v_k), k > 0$ where $G_1, \ldots, G_k$ are arbitrary sets of pairs of words, and $(u_0, v_0), \ldots, (u_k, v_k)$ are arbitrary pairs of words. The following are equivalent.*

1. *G is conjugate.*
2. *Each singleton redux of G (where all but one Kleene star is substituted with (ϵ, ϵ)) has a common witness z.*
3. *G has a common witness z.*

We prove that a rational relation is conjugate if and only if each of its constituent relation has a common witness. This provides a decision procedure for computing a common witness of a relation. The characterisation of conjugacy via common witness, together with this procedure, yields an algorithm for deciding conjugacy of rational relation.

C. Approximate Problems for Finite Transducers

Rational relations trivially extend rational functions, and sequential functions are strict subclass of rational functions. The class membership problems between the classes are known to be decidable. This includes the *functionality* problem [11], which asks whether a given a rational relation (by a transducer) has an equivalent rational function, and the *determinisation* problem [5], which asks whether a given rational function has an equivalent sequential function. We introduce approximate versions of these problems and show they are decidable as well.

The *approximate determinisation* problem w.r.t. distance d asks given a finite transducer recognising a function f, does there exists a sequential function g such that $d(f, g)$ is finite. For exact determinisation, determinisable finite transducers are characterised by the so called twinning property [5, 4], a pattern that requires that the delay between any two outputs on the same input must not increase when taking synchronised cycles of the transducer. We consider an approximate version of the twinning property (ATP), with no constraints on the delay, but instead requires that the output words produced on the synchronised loops are conjugate. It turns out that ATP is not sufficient to characterise approximately determinisable transducers, and an extra property is needed, the strongly connected twinning property (STP), which requires that the twinning property holds within strongly connected components of the finite transducer. We show that a transducer $\mathcal{T}$ is approximately determinisable (for Levenshtein family of distance) iff both ATP and STP hold and, if they do, we show how to approximately determinise $\mathcal{T}$. We also prove that ATP and STP are both decidable, and hence we get the following result.

▶ **Theorem 8** ([6]). *The approximate determinisation problem for rational functions w.r.t. Levenshtein family of distances are decidable.*

The *approximate functionality* problem asks whether a given rational relation R is *almost* a rational function, in the sense that $d(R, f)$ is finite for some rational function f, where $d(R, f)$ is now the supremum, for all $(u, v) \in R$, of $d(v, f(u))$. We get the following result as a consequence of computation of diameter of a rational relation.

▶ **Theorem 9** ([6]). *The approximate functionality problem for rational relations w.r.t. metrics given in Table 1 are decidable.*

We generalise the approximate determinisation problem to rational relations as well, which amounts to decide given a rational relation R, whether exists a sequential function f such that $d(R, f)$ is finite. We show that the characterisation for approximate determinisation of rational functions also holds for rational relations, and hence Theorem 8 also holds for rational relations as well.

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