Approximate Problems for Finite Transducers

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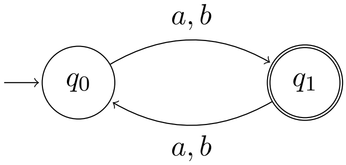
Finite State Transducer

- Machine that reads an input word and produces output word(s) using finite memory.
- Examples: spell checkers, grammatical tools.

Automaton vs. Transducer

Automaton

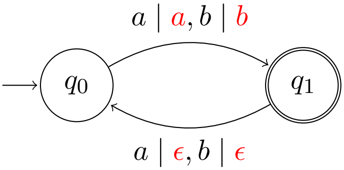
- Accepts a set of words.



Accepts odd length words.

Transducer

- Defines a relation over input-output words.

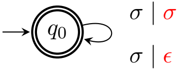


Outputs letters at odd positions.

$$aba \rightarrow aa$$

Rational Relations

- Rational relations are relations defined by transducers.

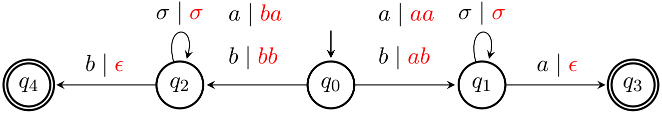


defines the relation $\{(u, v) \mid v \text{ is a subword of } u\}$.

$\sigma \in \{a, b\}$

Rational Functions

- Rational functions are functions defined by transducers.

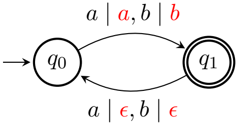


defines $\mathsf{f}_{\mathsf{last}} : \mathsf{u}\sigma \to \sigma\mathsf{u}$

$$\sigma \in \{a, b\}$$

Sequential Functions

- Sequential functions are functions defined by input-deterministic transducers.



- The function $\mathsf{f}_{\mathsf{last}} : \mathsf{u}\sigma \to \sigma\mathsf{u}$ is not sequential.

Recap: (Sub)classes of Transducers

$$\text{sequential functions} \subsetneq \text{rational functions} \subsetneq \text{rational relations}$$

- Rational relations - relations defined by transducers.
- Rational functions - functions defined by transducers.
- Sequential functions - functions defined by input-deterministic transducers.

Class Membership Problems

$$\text{sequential function} \xleftarrow{\text{determinisation}} \text{rational function} \xleftarrow{\text{functionality}} \text{rational relation}$$

Problem	Input	Question
Functionality	rational relation R	Is R a function?
Determinisation	rational function f	Is f sequential?

Class Membership Problems

$$\text{sequential function} \xleftarrow{\textsf{determinisation}} \text{rational function} \xleftarrow{\textsf{functionality}} \text{rational relation}$$

Problem	Result
Functionality	P [Choffrut, 1977, Weber and Klemm, 1995]
Determinisation	P [Schützenberger, 1975, Gurari and Ibarra, 1983]

- We study approximate versions of these problems.

Approximate Class Membership Problems

$$\text{sequential function} \xleftarrow{\text{apx. determinisation}} \text{rational function} \xleftarrow{\text{apx. functionality}} \text{rational relation}$$

Problem	Input	Question
Apx. Functionality	rational relation R	Is R close to a function?
Apx. Determinisation	rational function f	Is f close to a sequential function?

- Closeness can be measured using a notion of distance between functions/relations.

Distance between Words

- Edit distance between two words is the minimum number of edits required to rewrite one word to another.
- edits — substitutions, insertions, deletions, ...

Examples

$d(hello, yellow) = 2.$

$$\cancel{h}yellow$$

$$yellow$$

Common Edit Distances

Edit Distance	Permissible Operations
Hamming	letter-to-letter substitutions
Longest Common Subsequence	insertions and deletions
Levenshtein	insertions, deletions, and substitutions
Damerau-Levenshtein	insertions, deletions, substitutions and adjacent transpositions

Distance between Functions

- Let d be a distance on words. We can lift it to word-to-word functions.

$$d(f,g) = \begin{cases} \sup\{\, d(f(w), g(w)) \mid w \in dom(f)\} & \text{if } dom(f) = dom(g) \\ \infty & \text{otherwise} \end{cases}$$

Examples

- Consider functions $\mathsf{f}_{\mathsf{last}} : \mathsf{u}\sigma \to \sigma\mathsf{u}$ and $\mathsf{f}_{\mathsf{id}} : \mathsf{u}\sigma \to \mathsf{u}\sigma$

$$d(f_{\mathsf{last}}, f_{\mathsf{id}}) = 2 \text{ (w.r.t. Levenshtein distance).}$$

$$d(f_{\mathsf{last}}, f_{\mathsf{id}}) = \infty \text{ (w.r.t. Hamming distance).}$$

Distance between Functions

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- f and g are close if their distance $d(f,g)$ is finite.
- Edit distance between two rational functions is computable [Aiswarya et al., 2024].

Approximate Determinisation

$$\text{sequential function} \xleftarrow{\text{approx. determinisation}} \text{rational function}$$

- A rational function f is approximately determinisable w.r.t. a distance d if there exists a sequential function g such that $d(f, g)$ is finite.

Examples

- The function $\mathsf{f}_{\mathsf{last}} : \mathsf{u}\sigma \to \sigma\mathsf{u}$ is approx-determinisable w.r.t. Levenshtein.
- The function $\mathsf{f}_{\mathsf{id}} : \mathsf{u}\sigma \to \mathsf{u}\sigma$ is sequential and $d(f_{\mathsf{last}}, f_{\mathsf{id}})$ is finite.

Approximate Functionality

$$\text{rational functions} \xleftarrow{\text{approx. functionality}} \text{rational relations}$$

- A rational relation R is approximately functionalisable w.r.t. a distance d if there exists a rational function f such that $d(f, R)$ is finite.

Approximate Functionality

$$\text{rational functions} \xleftarrow{\text{approx. functionality}} \text{rational relations}$$

- A rational relation R is approximately functionalisable w.r.t. a distance d if there exists a rational function f such that
  - $dom(R) = dom(f)$ and
  - $\exists k$ s.t. on any input u, distance between $f(u)$ and $v \in R(u)$ is at most k,

$$\text{i.e., } \sup\{d(v, f(u)) \mid (u, v) \in R\} < \infty$$

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Examples

- Consider the rational relation $R = f_{ab} \cup f_{ba}$ where

$$f_{ab} : w \to (ab)^{|w|}$$

$$f_{ba} : w \to (ba)^{|w|}$$

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$$f_{ab} : w \to (ab)^{|w|} \qquad abab \cdots ab$$

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$$f_{ab} : w \to (ab)^{|w|} \qquad \cancel{a}bab \cdots aba$$

$$f_{ba} : w \to (ba)^{|w|} \qquad baba \cdots ba$$

- The function f_{ab} is rational and $d(f_{ab}, R) = 2$ w.r.t. Levenshtein.

Our Results

Problems \ Distances	Hamming	Levenshtein family
Approximate functionality	Decidable	Decidable
Approximate determinisation	Decidable	Decidable

Approximate Functionality: Characterisation

$$\mathsf{diff}_d(R) = \sup\{d(v_1, v_2) \mid \exists u \in dom(R), (u, v_1), (u, v_2) \in R\}$$

Lemma

A rational relation R is approximately functionalisable w.r.t. a distance d iff $\mathsf{diff}_d(R) < \infty$.

- $\mathsf{diff}_d(R)$ is computable for a given rational relation.

Distances / Problems	Hamming	Levenshtein family
Approximate functionality	Decidable	Decidable
Approximate determinisation	Decidable	Decidable

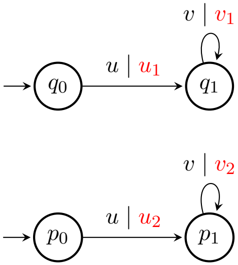
Exact Determinisation

- Extend automata subset construction.
- On any input, output the longest common prefix and keep the delay in memory.
- Construction terminates if the delay is finite.
- This is characterised using the *twinning property* of transducers [Choffrut, 1977].

$delay(u, v) = (u', v')$ s.t. $u = \ell u'$ and $v = \ell v'$ where ℓ is the longest common prefix of u and v.

Twinning Property

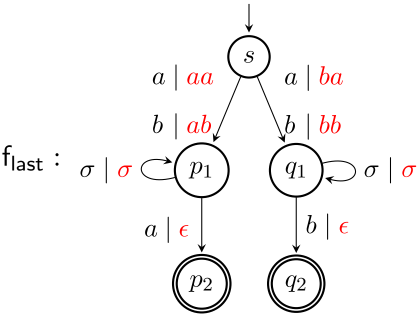
- A transducer $\mathcal{T}$ satisfies twinning property iff for all situations

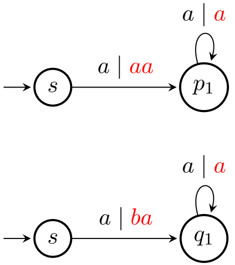


$$delay(u_1, u_2) = delay(u_1 v_1, u_2 v_2).$$

$delay(u, v) = (u', v')$ s.t. $u = \ell u'$ and $v = \ell v'$ where ℓ is the longest common prefix of u and v.

Twinning Property: Example



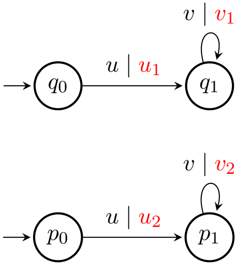


$$delay(aa, ba) \neq delay(aaa, baa).$$

$delay(u, v) = (u', v')$ s.t. $u = \ell u'$ and $v = \ell v'$ where ℓ is the longest common prefix of u and v.

Approximate Twinning Property (ATP)

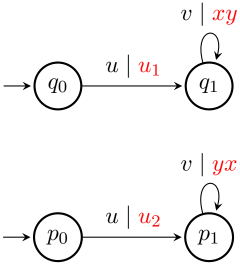
- A transducer $\mathcal{T}$ satisfies *approximate* twinning iff for all situations



v_1 and v_2 are conjugates

i.e., $\exists$ words x, y s.t. $v_1 = xy$ and $v_2 = yx$

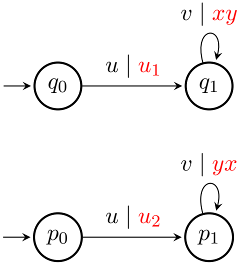
Why Conjugacy



$$xyxy \cdots xy$$

$$yxyx \cdots yx$$

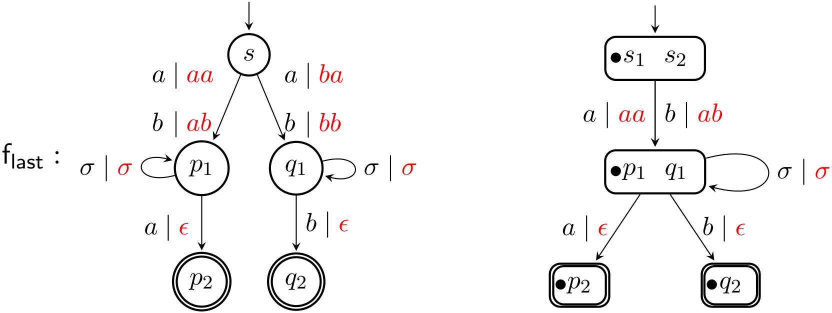
Why Conjugacy



$$\cancel{x}yxy\cdots xyx$$

$$yxyx\cdots yx$$

Approximate Determinisation: Construction



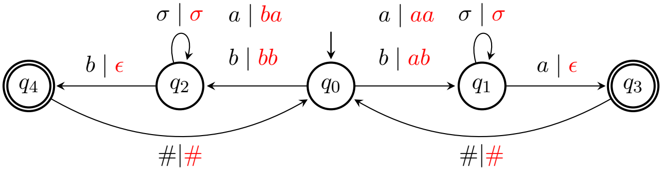
- Construction: extend automata subset construction. On any input, choose the output of the transducer with the smallest index.

Approximate Determinisation: Characterisation

- **ATP** is sufficient for certain subclassses of rational functions to be approx. determinisable.
  1. union of sequential transducers
  2. "concatenation" of sequential transducers

Approximate Determinisation: Characterisation

- **ATP** is sufficient for certain subclassses of rational functions to be approx. determinisable.
  1. union of sequential transducers
  2. ”concatenation” of sequential transducers
- **ATP** is not sufficient for rational functions to be approximately determinisable w.r.t. Levenshtein family of distances.



$\mathsf{f}^*_{\mathsf{last}} : \mathsf{u}_1\# \cdots \mathsf{u}_n\# \to \mathsf{f}_{\mathsf{last}}(\mathsf{u}_1)\# \cdots \mathsf{f}_{\mathsf{last}}(\mathsf{u}_n)\#$ is not approx. determinisable w.r.t. Levenshtein.

Approximate Determinisation: Characterisation

- For rational functions to be approximately determinisable w.r.t. Levenshtein, **ATP** + twinning property must hold within SCCs of the transducer (**STP**).

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Lemma

*A rational function given by a transducer $\mathcal{T}$ is approximately determinisable w.r.t. Levenshtein family of distances iff $\mathcal{T}$ satisfies **ATP** and **STP**.*

- Both **ATP** and **STP** are decidable properties for transducers.

Summary and Future Work

Problems \ Metrics	Hamming	Levenshtein family
Approximate determinisation	Decidable	Decidable
Approximate functionality	Decidable	Decidable
Approximate synthesis	Open	Undecidable

- Approximate synthesis asks given a rational relation R, does $\exists$ a sequential function *close* to some uniformiser of R?

Thank you :-)

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